Toric orbits as Lagrangian submanifolds

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The moment map:

We consider a $2n$-dimensional symplectic manifold (M^{2n}, ω) with a Hamiltonian action of the n-torus T^n. This is generated by a moment map

$$\Phi : M \to \mathbb{R}^n, \quad x \mapsto \big(H_1(x), \ldots, H_n(x)\big),$$

where the function $H_i : M \to \mathbb{R}$ generates the ith circle action via $\omega(\xi_i, \cdot) = dH_i$. (The vector field ξ_i is tangent to the orbits of the ith action.)

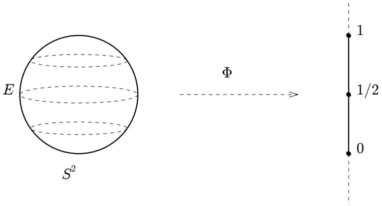

Figure : The simplest compact example is S^2 with S^1 acting by rotation about a vertical axis. Φ is the height function: $(x_1, x_2, x_3) \mapsto x_3$. The S^1-orbits are horizontal circles, the fibers $\Phi^{-1}(t)$ of the moment map.

The moment image:

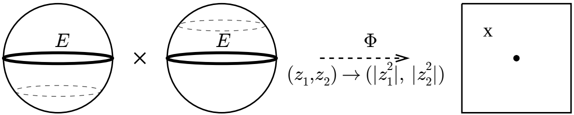

Figure : $S^2 \times S^2$ with the product action of $S^1 \times S^1$: the moment image $\Phi(S^2 \times S^2)$ is the square $[0,1] \times [0,1]$. $\Phi^{-1}(x)$ = circle orbit.

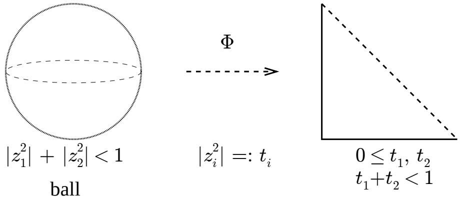

Figure : In polar coordinates (r, θ) on $\mathbb{R}^2$ with action $t \cdot (r, \theta) = (r, \theta + t)$ the moment map is $(r, \theta) \mapsto r^2$: hence $z \mapsto |z|^2 =: t$ for $z = re^{2\pi it} \in \mathbb{C}$. Thus the open ball $B^4 \subset \mathbb{C}^2 \equiv \mathbb{R}^4$ has moment image equal to a triangle .

Affine geometry:

The moment image $\Phi(M) = \Delta_M$ is a simple convex polytope. By Delzant's theorem, Δ_M determines the toric manifold (M, T^n, ω).

The shape of Δ_M depends on the chosen identification of T^n with the product $S^1 \times \cdots \times S^1$, together with the additive constants chosen for the Hamiltonians H_i. Thus $\Delta_M \subset \mathbb{R}^n$ is well defined modulo the action of the integral affine group $\mathrm{Aff}\,(n, \mathbb{Z}) \cong \mathrm{SL}\,(n, \mathbb{Z}) \times \mathbb{R}^n$ (symmetries of the torus). The relevant geometry is integral affine geometry.

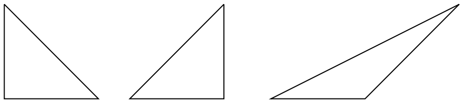

Figure : These triangles are all affine equivalent, describing the same toric manifold, namely the complex projective plane $\mathbb{C}P^2$.

Toric fibers:

The moment map (for closed M) is in fact the quotient map

$$\Phi : M \to M/T^n \equiv \Delta_M.$$

Each interior fiber $L_u := \Phi^{-1}(u), u \in \mathrm{Int}\Delta_M$ is a Lagrangian torus, i.e. $\omega|_{L_u} \equiv 0$. No two are equivalent under the action of the Hamiltonian group $\mathrm{Ham}(M, \omega)$ (the identity component of the symplectomorphism group) because they bound discs of different areas:

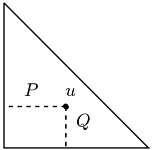

Figure : $\Phi^{-1}(P) \cong D^2(a) \times S^1$ and $\Phi^{-1}(Q) \cong D^2(b) \times S^1$, where $D^2(a), D^2(b)$ are discs whose areas a, b equal the affine lengths of the lines P, Q. (Note: ω restricts on $D^2 \times S^1$ to the pullback of the area form of the disc.) Also $L_u = S^1(a) \times S^1(b)$

Displaceability:

Which toric fibers are displaceable by a Hamiltonian isotopy? i.e. when is there $\phi \in \mathrm{Ham}(M)$ such that $\phi(L_u) \cap L_u = \emptyset$?

(Entov–Polterovich [EP09]): There is always at least one such fiber.

We can detect non-displaceable fibers using (variants) of Floer homology. In general, this is hard to define and needs auxiliary structures (certain 1-forms b).There has been much work on this by Cho, Cho–Oh, Fukaya–Oh–Ohta–Ono [FOOO10], Woodward, Wilson–Woodward [WW], Abreu–Macarini [AM13]...

FACT: If the Floer homology $HF(L_u, b)$ is defined and nonzero for some b, then L_u is non-displaceable.

FACT: $HF(L_u, b)$ can be nonzero only if u has at least two (and usually three) closest facets, where "closeness" is measured using affine distance d_{aff}. (More precisely, u must be a critical point of the Landau–Ginzburg potential.)

Examples:

The facet F with outward conormal η has equation.

$$F: \quad \eta \cdot u = \eta_1 x_1 + \eta_2 x_2 = \kappa$$

The affine distance $d_{aff}(u, F)$ between the point $u = (u_1, u_2)$ and F is $d_{aff}(u, F) = \kappa - (\eta_1 u_1 + \eta_2 u_2)$.

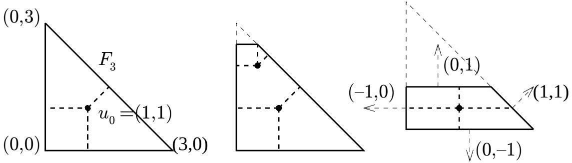

Figure : On the left is $\mathbb{C}P^2$ with a (unique) non-displaceable fiber at the center of gravity u_0 of Δ. Note that $d_{aff}(u_0, F_1) = 1 = d_{aff}(u_0, F_3)$. The other two figures are one point blow-ups, a small one in the middle with two non-displaceable points, and a large blow-up on the right with one. These four heavy dots mark the only points in these examples where the Floer homology does not vanish.

Displacing fibers

A probe P is a line segment in $\Delta \subset \mathbb{R}^n$

- starting at a point q in the interior of a facet F and
- whose direction vector ν can be completed to an integral basis for $\mathbb{R}^n$ by adding vectors parallel to F.

FACT: A point $u \in \mathrm{Int}\Delta$ is displaceable (i.e. the corresponding toric fiber L_u can be moved off itself by a Hamiltonian isotopy) if it lies less than halfway along a probe: cf. McDuff [Mc11i].

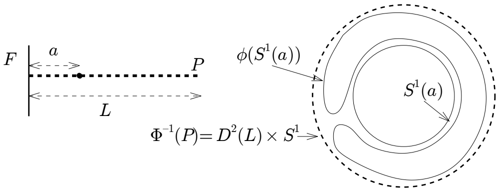

Figure : If P has affine length L, $\Phi^{-1}(P) = D^2(L) \times S^1 \subset M$, with symplectic form pulled back from the disc, while $\Phi^{-1}(u) = D^2(a) \times S^1$ if u is a distance a along the probe. So if $a < L/2$ we can displace $S^1(a)$ to $\phi(S^1(a))$ inside $D^2(L)$ by an area-preserving map, and then extend this deformation to M.

Some Probes and non Probes :

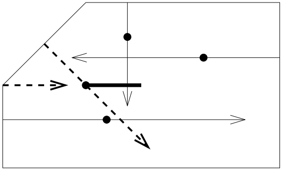

Figure : probes are solid lines with arrows, non probes are dotted: either the direction or starting point is wrong.

In particular, the downward diagonal line has direction $(1, -1)$, and it starts on a facet with direction $(1, 1)$: but these two vectors have determinant $= 2$ so do *not* form an integral basis.
All the points inside this figure except for those on the very heavy horizontal line are displaceable by probes. Those on the line are not: in fact they all have nontrivial Floer homology [FOOO12].

Gonzalez–Woodward [GW12] interpret the non displaceable points detected by probes in terms of the minimal model program.

Floer homology and Probes:

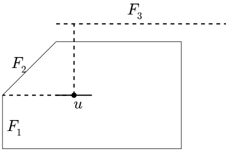

Figure : Floer homology is calculated by an iterative construction: since the points on the horizontal are equidistant from two nearest parallel lines we can look at the next closest facets, adding a ghost facet to make each such point u equidistant from three facets (here F_1, F_2, F_3.) Abreu–Macarini explain this in terms of symplectic reduction.

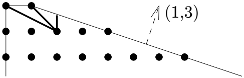

Figure : A point whose Floer homology vanishes, but is inaccessible by probes: the potential probes that reach this point either start at a vertex or (as in the case of the vertical line) have a bad direction. This point can be displaced by an extended probe; cf. Abreu–Borman–McDuff.

Some Open Problems:

Problem 1: A monotone toric manifold, scaled so that $[\omega] = c_1$, has a unique interior integral point. It is the unique point with nontrivial Floer homology. Is this the only non displaceable point? There is a known $n = 6$-dimensional example where this point cannot be displaced by probes. cf. McDuff [Mc11ii]

Problem 2: Probes (and Floer invariants) can also be used to investigate similar questions in toric orbifolds. These tend to have fewer displaceable fibers and a richer set of invariants.

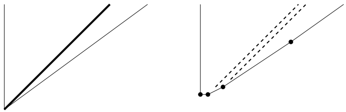

Figure : On the left the A_3-singularity: points on the dark rays have nontrivial Floer invariants, and the rest are displaceable by (extended) probes. On the right the resolved singularity: here all Floer invariants vanish, but no ways are known to displace the points on the dotted lines.

A few References:

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