Smooth Kuranishi atlases

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Contents:

This is a talk about my ongoing project with Katrin Wehrheim [MW1,2] to clarify the construction of the virtual moduli cycle in the Gromov–Witten context. Basic question: how can one count closed J-holomorphic curves in a symplectic manifold in a way that is independent of choices?

The talk will be divided into several parts.

Part 1: Explanation of the problem.

Part 2: Kuranishi atlases.

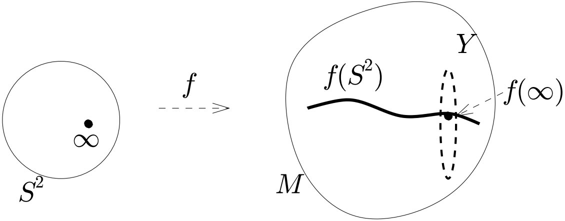
Part 3: Building charts with isotropy.

Counting J-holomorphic spheres:

(M^{2n}, ω, J) is a symplectic manifold with an ω-tame almost complex structure J. We want to count the number of J-holomorphic maps $f : (S^2, j) = (\mathbb{C} \cup \infty, j) \to (M, J)$, with $f_*([S^2]) = A \in H_2(M, \mathbb{Z})$ (modulo parametrization) that satisfy certain homological constraints: e.g. if $Y \subset M$ is a submanifold, count

$$\#\left(\left\{ f \mid 0 = \overline{\partial}_J f =: \tfrac{1}{2}(df + J \circ f \circ j),\ f(\infty) \in Y \right\} \big/ G_\infty \right)$$

where G_∞ is the group of reparametrizations $z \mapsto az + b$ fixing ∞.



The virtual fundamental class VFC:

More abstractly, we would like to show that the space of solutions X to the Fredholm equation $\overline{\partial}_J f = 0,\ f_*([S^2]) = A$ has a homology class

$$[X]^{\mathrm{vir}} \in H_d(X)$$

(in the dimension d of the index of the Fredholm operator).

Then the count would just be the intersection number $\mathrm{ev}_*([X]^{\mathrm{vir}}) \cdot_M [Y]$ where ev is the evaluation map $\mathrm{ev}(f) = f(\infty)$.

In the semi-positive case, one can often choose J so that X is a manifold of dimension d; if noncompact, it often still has “boundary” of codimension ≥ 2, so that $\mathrm{ev} : X \to M$ represents a well defined homology class (independent of choice of J).

Problems with the naive (geometric) approach: 5.

- Nonregular J: Even if J is generic, the space of solutions $X := \mathcal{M}_{0,1}(A, J)$ to the Fredholm equation $\overline{\partial}_J f = 0,\ f_*([S^2]) = A$ is not usually a manifold of the right dimension (i.e. equal to the Index of $\overline{\partial}_J$).
- Existence of isotropy: Some solutions have an internal symmetry (e.g. $f : z \mapsto z^2$ is invariant under the action of $\mathbb{Z}/2\mathbb{Z}$ by $z \mapsto -z$). So $X := \mathcal{M}_{0,1}(A, J)$ has to be given some kind of orbifold structure.
- Lack of compactness: Usually one must compactify the solution space by adding nodal curves that then must be glued together, greatly complicating the required analysis.
- Lack of smoothness: One must use Banach space of maps $f : S^2 \to M$ (e.g. $W^{k,p}$-maps) to do Fredholm analysis, but the reparametrization group does not act differentiably on such a space.

In 1990s, great progress with these issues by Li-Tian, Fukaya–Ono and others (Ruan, Siebert, . . .). Emphasis on gluing analysis; sketchy treatment of smoothness/topological issues in constructing charts, VFC.

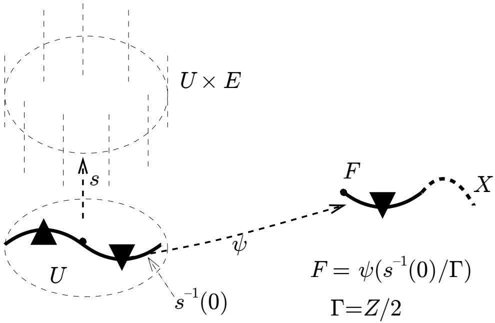
Recent new developments:

- Polyfolds: Hofer–Wysocki–Zehnder (2006 –now) series of papers completely redoing the analytic foundations. Gromov–Witten preprint: arxiv 1107.2097
- Smooth Kuranishi atlases: McDuff–Wehrheim, arXiv 1208.1340: recast the basic definitions of FO, FOOO in more categorical terms; redid the topological aspects of the proof in a special case (no nodes, no isotropy).
- Detailed treatment of Kuranishi structures: Fukaya–Oh–Ohta–Ono, arXiv 1209.4410; *many* more details of the general case, including smoothness of gluing.
- New, wholistic approach using "Donaldson divisors": Ionel–Parker, arXiv 1304.3472; more geometric approach initially suggested by Cieliebak–Mohnke.
- Reworking of the Tian approach using "virtual manifolds": Chen–Li–Wang, arXiv 1306.3276. new way of dealing with the nonsmooth action.

Finite dimensional reductions [FO], [FOOO], [MW], [CLW]: 7.

The moduli space X can be locally modelled by a finite dimensional reduction (U, E, Γ, s, ψ), where

- the domain U is a smooth manifold, the obstruction space $E \cong \mathbb{R}^n$;
- the isotropy group Γ is a finite group acting diagonally on $U \times E$;
- the section $s : U \to E$ is induced by $\overline{\partial}_J$, and is Γ-equivariant;
- the footprint map $\psi : s^{-1}(0) \to X$ induces a homeomorphism ${s^{-1}(0)}/{\Gamma} \to F$, where $F \subset X$ is open.



Kuranishi atlases [MW1]:

Let X be a compact, metrizable space. A Kuranishi chart for X with footprint F is a tuple $\mathbf{K} = (U, E, \Gamma, s, \psi)$ with $F = \operatorname{im}\psi$.

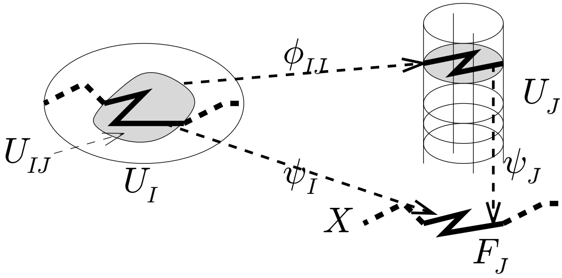
- A covering family of basic charts for X is a finite collection $(\mathbf{K}_i)_{i=1,\dots,N}$ of Kuranishi charts with $X = \bigcup_{i=1}^N F_i$.
- Transition data for $(\mathbf{K}_i)_{i=1,\dots,N}$ is a collection of charts $(\mathbf{K}_J)_{J\in\mathcal{I}_\mathcal{K},|J|\geq 2}$ and coordinate changes $(\widehat{\Phi}_{IJ})_{I,J\in\mathcal{I}_\mathcal{K}, I\subsetneq J}$ as follows:
  1. $\mathcal{I}_\mathcal{K}$ is the set of subsets $I \subset \{1,\dots,N\}$ s.t. $F_I := \bigcap_{i\in I} F_i \neq \emptyset$;
  2. the sum chart $\mathbf{K}_J$ has footprint $F_J = \bigcap_{i\in J} F_i$ and additive obstruction space $E_J \cong \oplus E_{i\in J}$; (in GW case must be built using some analysis)
  3. all charts have the same dimension $d := \dim U_I - \dim E_I = \operatorname{Ind}(\overline{\partial}_J)$ (so $\dim U_I \leq \dim U_J$ if $I \subset J$.)
  4. $\widehat{\Phi}_{IJ}$ is a coordinate change $\mathbf{K}_I \to \mathbf{K}_J$ for every $I, J \in \mathcal{I}_\mathcal{K}$ with $I \subsetneq J$. (to be explained)

A Kuranishi atlas $\mathcal{K}$ on X consists of $\left(\mathbf{K}_J, \widehat{\Phi}_{IJ}\right)$ as above satisfying the cocycle condition. (to be explained)

Coordinate changes I (no isotropy):

If $I \subset J$ then $F_I \supset F_J$ and there is a coordinate change $\mathbf{K}_I \to \mathbf{K}_J$, consisting of a restriction $\mathbf{K}_I \to \mathbf{K}_I\big|_{U_{IJ}}$ followed by an inclusion $\mathbf{K}_I\big|_{U_{IJ}} \to \mathbf{K}_J$ induced by an embedding $\phi_{IJ} : U_{IJ} \to U_J$, where

- $U_{IJ} \subset U_I$ (in grey) intersects the zero set $s_I^{-1}(0)$ in $\psi_I^{-1}(F_J)$;
- the (grey) image of $\phi_{IJ} : U_{IJ} \to U_J$ is a smooth submanifold with "nice" normal bundle;
- there is an associated linear inclusion $\widehat{\phi}_{IJ} : E_I \to E_J$;
- obvious compatibility conditions with footprint maps and sections; (eg for $x \in \psi_I^{-1}(F_J)$, we have $\psi_J \circ \phi_{IJ}(x) = \psi_I(x) \in F_J$)



Cocycle condition and Kuranishi category (no isotropy) : 10.

Aim: each atlas $\mathcal{K}$ should have domain and bundle categories with functor $\mathrm{proj} : \mathbf{E}_{\mathcal{K}} \to \mathbf{B}_{\mathcal{K}}$, where $\mathrm{Obj}_{\mathbf{B}_{\mathcal{K}}} = \bigcup_I U_I$ and the morphisms $\mathrm{Mor}_{\mathbf{B}_{\mathcal{K}}} = \bigcup_{I \subseteq J} U_{IJ}$ are defined by the coordinate changes.

- The cocycle condition ensures that composition is possible, i.e. if $\phi_{JK} \circ \phi_{IJ}(x)$ is defined for $x \in U_{IJ}$ so is $\phi_{IK}(x)$ and they are equal.
- In practice, one can only construct weak atlases with the weak cocycle condition, i.e. whenever both maps are defined, they are equal.
- We also want the realization $|\mathcal{K}| := \bigcup_I U_I / \sim$ of $\mathbf{B}_{\mathcal{K}}$ to have good properties eg Hausdorff, a well-behaved metric...

Proposition ([MW1], §6): *A weak Kuranishi atlas can be tamed to form a Kuranishi atlas $\mathcal{K}$ with a "nice" realization $\big|\mathcal{K}\big|$.*

From (weak) Kuranishi category to VFC (no isotropy): 11.

[MW1] Theorem B. *Let $\mathcal{K}$ be an oriented, d-dimensional, weak, additive Kuranishi atlas with trivial isotropy groups on a compact metrizable space X. Then $\mathcal{K}$ determines a cobordism class of smooth, oriented, compact manifolds, and an element $[X]^{vir}_{\mathcal{K}}$ in the Čech homology group $\check{H}_d(X;\mathbb{Q})$. Both depend only on the cobordism class of $\mathcal{K}$.*

The class $[X]^{vir}_{\mathcal{K}}$ is represented by the zero set $Z(s+\nu)$ – which is a manifold – of a transverse perturbation $|s+\nu|$ of the section $|s|$ of $|\mathrm{proj}| : \left|\mathbf{E}_{\mathcal{K}}\right| \to \left|\mathcal{K}\right|$.

The perturbation ν is tricky to construct: there is an inclusion $\iota_X : X \to \left|\mathcal{K}\right|$ with image equal to $Z(s)$, but $Z(s)$ does NOT have compact neighbourhood in $\left|\mathcal{K}\right|$; so need to work to ensure that $Z(s+\nu)$ is compact.

Kuranishi atlases with isotropy [MW2], :

Recall: An atlas is: $\mathcal{K} = \big(\mathbf{K}_I, \widehat{\Phi}_{IJ}\big)$ where $(\mathbf{K}_i)$ is a covering family, $\mathbf{K}_I$ are sum charts, and each $\widehat{\Phi}_{IJ} : \mathbf{K}_I \to \mathbf{K}_J$ is a coordinate change over F_J with cocycle condition for $I \subset J \subset K$. When there is no isotropy, $\widehat{\Phi}_{IJ}$ is a restriction to $U_{IJ} \subset U_I$ followed by an "inclusion" $\phi_{IJ} : U_{IJ} \to U_J$. With isotropy, the picture changes a little:

For $I \subset J$, the group $\Gamma_J := \prod_{i \in J} \Gamma_i$ of $\mathbf{K}_J$ splits as $\Gamma_J = \Gamma_I \times \Gamma_{J \setminus I}$.
The coordinate change is given by a subset $\widetilde{U}_{IJ} \subset U_J$ where

- $\widetilde{U}_{IJ}$ is Γ_J invariant, where $\Gamma_{J\setminus I}$ acts freely;
- the quotient ${}^{\widetilde{U}_{IJ}}/_{\Gamma_{J\setminus I}}$ can be identified Γ_I-equivar. with $U_{IJ} \subset U_I$; we get equivariant covering map $\rho_{IJ} : \widetilde{U}_{IJ} \to U_{IJ}$ that intertwines s, ψ.
- the category $\mathbf{B}_{\mathcal{K}}$ has morphisms $\widetilde{U}_{IJ} \times \Gamma_I$ with

$$s \times t : \widetilde{U}_{IJ} \times \Gamma_I \ni \big(x, \gamma_I\big) \mapsto \big(\gamma_I^*(\rho_{IJ}(x)), x\big) \in U_I \times U_J,$$

(these are morphisms from a subset of U_I into U_J, coming from the group actions and coord changes)

From the intermediate category to the VFC:

The intermediate chart $\underline{\mathbf{K}}_I$ has no isotropy, but has orbifold domain and bundle:

$$\underline{U}_I := {}^{U_I}/_{\Gamma_I}, \quad \underline{U_I \times E_I} := {}^{U_I \times E_I}/_{\Gamma_I}; \quad \pi_I : U_I \to \underline{U}_I.$$

A Kur. atlas $\mathcal{K}$ has an intermediate domain category $\underline{\mathbf{B}}_{\mathcal{K}}$ where

$$\mathrm{Obj}_{\underline{\mathbf{B}}_{\mathcal{K}}} = \bigcup_I \underline{U}_I, \quad \mathrm{Mor}_{\underline{\mathbf{B}}_{\mathcal{K}}} = \bigcup_{I \subset J} \underline{U}_{IJ}, \text{ where } \underline{U}_{IJ} = \pi_J(\widetilde{U}_{IJ}) \subset \underline{U}_J.$$

- The functor $\mathbf{B}_{\mathcal{K}} \to \underline{\mathbf{B}}_{\mathcal{K}}$ induces a proper map on objects and morphisms. (inverse images of compact sets are compact).
- The previous work for the "no isotropy" case applies to $\underline{\mathbf{B}}_{\mathcal{K}}$ and then lifts to $\mathbf{B}_{\mathcal{K}}$. So we can construct tamings and (multi)sections as before.
- Thus get VFC represented by a weighted branched manifold, well defined up to cobordism; completing the abstract theory.

Charts for the 2, 3-football orbifold:

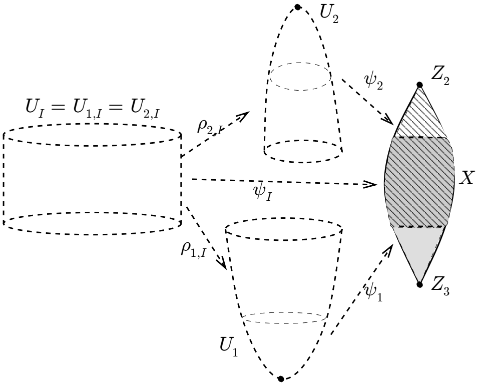


Figure : X is S^2 with two orbifold points of orders 2, 3. There are two basic charts $(U_1, \mathbb{Z}_3), (U_2, \mathbb{Z}_2)$ with images discs $\underline{U}_i \subset X$, and one transition chart $(U_I, \mathbb{Z}_2 \times \mathbb{Z}_3)$ (where $I = \{1, 2\}$) with image the annulus $\underline{U}_I = \underline{U}_1 \cap \underline{U}_2$.

Gromov–Witten case [MW2]: set up for basic chart :

$$X = \left\{f : S^2 \to (M,J) \mid \overline{\partial}_J f = 0,\ f_*([S^2]) = A\right\}/_{G_\infty}.$$

To make chart at $[f_0] \in X$ with $\Gamma = \mathrm{Stab}([f_0])$, must

- Stabilize domain by adding two marked points $0, 1$ and fix parametrization f_0 by fixing $f_0(0), f_0(1)$; This is achieved via slicing conditions: i.e. Choose $Q^{2n-2} \subset M$ transverse to $\mathrm{im}\, f_0$, and require $f_0(0), f_0(1) \in Q$;
- Extend $v^0 := 0, v^1 = 1$ to a (minimal) Γ-invariant tuple $\vec{v} = (v^0, \dots, v^{L-1})$;
- Choose Γ-invariant set of disjoint disc nbhds $D^{f_0}(v^\ell) \subset S^2$ s.t. $f_0^{-1}(Q) \cap D^{f_0}(v^\ell) = \{v^\ell\}\ \forall \ell$;
- Choose obstruction space $E_0 \subset C^\infty\text{-sect}(\mathcal{H})$ where $\mathcal{H} \to S^2 \times M$ is bundle with $\mathcal{H}_{(z,x)} = \mathrm{Hom}_J^{0,1}(T_z S^2, T_x M)$ s.t. E_0 covers $\mathrm{coker} D_{f_0}(\overline{\partial}_J)$
- This gives a "geometrically defined" obstruction space $\vec{E} := \oplus_{\gamma \in \Gamma} E_0^\gamma$, the sum of $|\Gamma|$ copies of E_0. ("Abstract" obstruction spaces do not usually transform C^1 smoothly under coordinate changes.)

Gromov–Witten case: a basic chart II:

- Given $\vec{w} = (w^0, \dots, w^{L-1}) \in (S^2)^L$ define $\phi_{\gamma,\vec{w}} \in G_\infty$ to be the unique map s.t. $0 \mapsto w^{\gamma(0)}, 1 \mapsto w^{\gamma(1)}$.
- Define $U \subset \vec{E} \times (S^2)^L \times \mathrm{nbhd}(f_0)$ by "Fredholm stabilization":

$$U : \;=\; \Big\{(\vec{\nu}, \vec{w}, f) \;\Big|\; \bar{\partial}_J f = \sum_\gamma (\phi^{-1}_{\gamma,\vec{w}})^* \nu^\gamma \Big|_{\mathrm{graph} f},$$

$$\text{(normalization)}\; w^0 = 0, w^1 = 1, \quad \text{(slicing)}\; w^\ell = f^{-1}(Q) \cap D^{f_0}(v^\ell) \Big\}$$

- Action of Γ: $(\vec{\nu}, \vec{w}, f) \mapsto \gamma^*(\vec{\nu}, \vec{w}, f) = (\gamma^*(\vec{\nu}), \gamma^*(\vec{w}), f \circ \phi_{\gamma,\vec{w}})$, where $\big(\gamma^*(\vec{\nu})\big)^\ell = \nu^{\gamma(\ell)} =: (\gamma \cdot \vec{\nu})^\ell$,
$\big(\gamma^*(\vec{w})\big)^\ell = \phi^{-1}_{\gamma,\vec{w}}(w^{\gamma(\ell)}) = \phi^{-1}_{\gamma,\vec{w}}((\gamma \cdot \vec{w})^\ell)$. (This preserves slicing conditions and normalization $\gamma^*(\vec{w}) = (0, 1, \dots)$: you can't just permute the tuple $\vec{w}$ since we have normalized.)
- The chart is $(U, \vec{E}, \Gamma, s, \psi)$ where $s(\vec{\nu}, \vec{w}, f) = \vec{\nu} \in \vec{E}$, $\psi(\vec{\nu}, \vec{w}, f) = [f]$.

Gromov–Witten case: a transition chart:

Given $(U_i, \vec{E}_i, \Gamma_i, s_i, \psi_i)$, pick $i_0 \in \{1,2\}$ and define $\mathbf{K}_I$ for $I = \{1,2\}$ with domain i_0-normalized; so $U_I \subset \mathcal{W}_{I,i_0}$ where

$$\mathcal{W}_{I,i_0} = \{(\underline{\vec{\nu}}, \underline{\vec{w}}, f) \in \vec{E}_1 \times \vec{E}_2 \times (S^2)^{L_1+L_2} \times \mathrm{nbhd}(f_0) \mid w^0_{i_0} = 0, w^1_{i_0} = 1\}.$$

- $U_I := \Big\{(\underline{\vec{\nu}}, \underline{\vec{w}}, f) \mid \overline{\partial}_J f = \sum_{i=1,2,\gamma\in\Gamma_i} (\phi^{-1}_{\gamma,\vec{w}_i})^* \nu_i^\gamma\Big|_{\mathrm{graph} f},\ f(w_i^\ell) \in Q_i\Big\}$

  (here $U_I \subset \mathcal{W}_{I,i_0}$; also must specify the ordering of the tuple $\vec{w}_j$, $j \neq i_0$, more carefully)
- Action of Γ_I depends on i_0: with $i_0 = 1$
  - $\gamma \in \Gamma_1$ acts by perm. and reparametrization (to preserve normalization)

    $$\gamma^*(\vec{\nu}_1, \vec{\nu}_2, \vec{w}_1, \vec{w}_2, f) = (\gamma\cdot\vec{\nu}_1, \vec{\nu}_2, \phi^{-1}_{\gamma,\vec{w}_1}(\gamma\cdot\vec{w}_1), \phi^{-1}_{\gamma,\vec{w}_1}(\vec{w}_2), f\circ\phi_{\gamma,\vec{w}_1}),$$
  - $\gamma \in \Gamma_2$ acts just by permutation

    $$\gamma^*(\vec{\nu}_1, \vec{\nu}_2, \vec{w}_1, \vec{w}_2, f) = (\vec{\nu}_1, \gamma\cdot\vec{\nu}_2,\ \vec{w}_1, \gamma\cdot\vec{w}_2, f),$$
- the projection $\rho_{1,I} : U_I \supset \widetilde{U}_{1,I} \to U_1$ is the forgetful map $(\vec{\nu}_1, \vec{\nu}_2, \vec{w}_1, \vec{w}_2, f) \mapsto (\vec{\nu}_1, \vec{w}_1, f)$ (and for $i = 2$ is forgetful map plus renormalization. i.e. All elements of the construction are very natural.)

Further directions:

- The previous slides attempt to explain how to construct a Kuranishi atlas near the parts of X represented by curves with smooth domains.
- It remains to deal with nodal curves. Instead of using a fancy gluing theorem that would give a smooth structure to the domains U near a nodal curve, we will use the standard theorem in [Mc-Sal] that gives continuity in the gluing parameters a but with evaluation maps depending C^1 on a. This is enough to give a Kuranishi atlas whose domains are stratified smooth. [Details still to be written, should be pretty straightforward]
- An interesting special case is when there is a regular J (e.g. for a space of genus zero stable maps into $\mathbb{C}P^n$.) Then there is an orbifold Kuranishi atlas (i.e. all E trivial) [Details still to be written]
- Potential generalizations: to curves in manifolds with S^1 action; to curves with intrinsic Lie group symmetries (e.g. Hamiltonian Floer homology with time independent Hamiltonian [needs extra work to deal with the boundary]), to Gromov–Witten invariants for symplectic orbifolds . . . [none of this done at all]

A few recent References:

[CLW13] Bohui Chen, An-Min Li, and Bai-Ling Wang, Virtual neighborhood technique for pseudo-holomorphic spheres, arXiv:1306.3276.

[FOOO12] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, Technical detail on Kuranishi structure and Virtual Fundamental Chain, arXiv:1209.4410.

[HWZ4] H. Hofer, K. Wysocki, and E. Zehnder, Applications of Polyfold theory I: The Polyfolds of Gromov–Witten theory, arXiv:1107.2097.

[IP] E. Ionel and T. Parker, A natural Gromov–Witten fundamental class, arXiv:1302.3472

[Mc-Sal] D. McDuff and D.A. Salamon, *J-holomorphic curves and symplectic topology*, Colloquium Publications **52**, American Mathematical Society, Providence, RI, (2004), 2nd edition (2012).

[MW1] D. McDuff and K. Wehrheim Smooth Kuranishi atlases without isotropy, arXiv:1208.1340.

[MW2] D. McDuff and K. Wehrheim Smooth Kuranishi atlases with isotopy, in preparation