

# A Glimpse into Symplectic Geometry

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## §1 Introduction

Over the past 15 years symplectic geometry has developed its own identity, and can now stand alongside traditional Riemannian geometry as a rich and meaningful part of mathematics. The basic definitions are very natural from a mathematical point of view: one studies the geometry of a skew-symmetric bilinear form  $\omega$  rather than a symmetric one. However, this seemingly innocent change of symmetry has radical effects. For example, one dimensional measurements vanish since  $\omega(v, v) = -\omega(v, v)$  by skew-symmetry. Hence symplectic geometry is an essentially 2-dimensional geometry that measures the *area* of a complex curve instead of the *length* of a real curve.

In this article, we will try to explain the basic ideas of symplectic geometry. Here are some of the features that distinguish it from more traditional geometries.

- The theory has two faces. There are two kinds of geometric subobjects in a symplectic manifolds, hypersurfaces and Lagrangian submanifolds that appear in dynamical constructions, and even-dimensional symplectic submanifolds that are closely related to Riemannian and complex geometry. As we shall see, the analog of a geodesic in a symplectic manifold is a two-dimensional surface called a  $J$ -holomorphic curve.
- Although all its concepts are initially expressed in the smooth category (for example, in terms of differential forms), in some intrinsic way that is not yet well understood they do not really depend on derivatives. There is a notion

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of the symplectic capacity of a subset  $A$  that is continuous with respect to the Hausdorff distance function on sets and has the property that a diffeomorphism is (anti-) symplectic if and only if it preserves capacity. Thus symplectic geometry is essentially topological in nature. Indeed one often talks about symplectic topology.

- Darboux’s theorem says that locally all symplectic spaces are the same. Thus the only invariants that distinguish one symplectic space from another are global. On the other hand, the lack of local invariants makes it possible for there to be many automorphisms of the local structure. Indeed any smooth function on a symplectic manifold generates a flow on the manifold that preserves the symplectic structure.

- There is an interesting phenomenon that might be called a *local to global principle*: statements that hold locally often have analogs that are valid globally. This can be interpreted in various ways. The local statement might be linear, or something valid for short periods of time, or something valid in a small open subset of a manifold on which the geometry is standard. Correspondingly, the global statement would be nonlinear, or valid for all time, or valid on all manifolds.

- Symplectic structures first arose in the Hamiltonian formulation of the theory of classical mechanics and this tight interconnection with physics has persisted ever since. A longstanding mystery in mathematics that is especially striking now is the extraordinary power of ideas from physics, specially duality theories and string theory, which lead to unexpected and so far little understood symmetries: see Vafa.... In particular mirror symmetry applies to certain special symplectic manifolds, and interchanges two quite different aspects of these spaces. One of these aspects (quantum cohomology) arises immediately from the properties of the “symplectic geodesics” ( $J$ -holomorphic curves) although the other is somewhat more obscure. However, the whole framework in which mirror symmetry is now discussed is that of symplectic geometry. This should not be too much of a surprise since this geometry has involutive symmetries built into its foundations. (As a simple example, there is a natural symplectic structure on the sum  $E \oplus E^*$  of a vector space and its dual.)

In this article we will first give some of the basic definitions in the subject, trying to bring out both the “complex” and the “Lagrangian” aspects of the geometry. Most of the most profound recent results involve both of these aspects. We will illustrate this by means of a few examples. There are rather few references mentioned here: those interested can find much more complete listings in [Du], [H], [M2], [MS], [V]. The book [MS] contains useful background material and [M1] is another survey article.

## Basic notions

Let  $M^{2n}$  be a smooth manifold without boundary. A symplectic structure  $\omega$  on  $M$  is a closed ( $d\omega = 0$ ), nondegenerate ( $\omega^n = \omega \wedge \dots \wedge \omega \neq 0$ ) smooth 2-form. Thus the intrinsic measurements that one can make on a symplectic manifold are 2-dimensional, i.e. if  $S$  is a little piece of 2-dimensional surface then one can measure

$$\int_S \omega = \text{area}_\omega S.$$

By Stokes' theorem, the closedness of  $\omega$  is equivalent to saying that this integral does not change when one deforms  $S$  keeping its boundary fixed. The non-degeneracy condition is equivalent to the fact that  $\omega$  induces an isomorphism between the vector fields  $X$  on a manifold and the space of 1-forms via the correspondence:

$$\begin{array}{ccc} T_x M & \xrightarrow{\cong} & T_x^* M \\ X & \mapsto & \iota_X \omega = \omega(X, \cdot) \\ \text{vector fields} & & \text{1-forms} \end{array}$$

**First Example.** The linear form  $\omega_0 = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$  on Euclidean space  $\mathbf{R}^{2n}$ . In this case, the above isomorphism is given explicitly by the formulae

$$\begin{aligned} X = \frac{\partial}{\partial x_j} & \mapsto \iota_X \omega_0 = dy_j \\ Y = \frac{\partial}{\partial y_j} & \mapsto \iota_Y \omega_0 = -dx_j. \end{aligned}$$

In Riemannian geometry one identifies the tangent space  $T_x \mathbf{R}^{2n}$  of vectors and the cotangent space  $T_x^* \mathbf{R}^{2n}$  of covectors (or 1-forms) by making the following identifications:

$$\frac{\partial}{\partial x_j} \equiv dx_j, \quad \frac{\partial}{\partial y_j} \equiv dy_j.$$

The isomorphism given by a symplectic form differs from this by a rotation through a quarter turn. If one pairs the coordinates  $x_j, y_j$  thinking of them as the real and imaginary parts of a complex coordinate  $z_j = x_j + iy_j$ , then this quarter turn corresponds to multiplication by  $i$ . Below, we call it  $J_0$ .

**Other Examples.** Every Kähler manifold  $(M, J, g)$  has a symplectic structure  $\omega_J$ . Recall that a Kähler manifold  $M$  is made from pieces of complex Euclidean space  $\mathbf{C}^n$  that are patched by holomorphic maps. One adds a metric  $g$  to this complex manifold and then defines the symplectic form  $\omega_J$  by setting

$$\omega_J(x, y) = g(Jx, y).$$

(For this to work,  $g$  must be compatible with  $J$  in a rather strong sense:  $J$  has to be parallel with respect to the Levi-Civita connection in order for  $\omega_J$  to be closed. Not all complex manifolds can be given a Kähler structure.) Kähler manifolds are a very important class of symplectic manifolds. However they have some properties that are not shared by arbitrary symplectic manifolds and one should think of the latter as more flexible versions of Kähler manifolds. It is

still not understood exactly how much arbitrary symplectic manifolds can differ from Kähler ones.

An opposite kind of example (relating to the dynamic aspects of symplectic geometry) is given by the cotangent bundle  $T^*X$  of an arbitrary smooth manifold  $X$ . Because  $T^*X$  is the set of all 1-forms on  $X$ , it carries a universal 1-form  $\lambda_{can}$ . (This can be written as  $\sum_i p_i dq_i$ , in terms of local coordinates  $(p_i, q_i)$  on  $T^*X$ , where  $q_1, \dots, q_n$  are coordinates on  $X$  and the  $p_i$  are the corresponding momentum coordinates in the fibers  $T_q^*X$ .) The canonical symplectic form  $\Omega_{can}$  on  $T^*X$  is then simply  $-d\lambda_{can}$ .  $\square$

**Submanifolds** As mentioned above there are several interesting kinds of submanifolds in a symplectic manifold  $(M, \omega)$ . First of all there are submanifolds  $Q$  on which the restriction of  $\omega$  is nondegenerate. These are called *symplectic* submanifolds, and, of necessity, they are even dimensional. For example any complex submanifold of a Kähler manifold is symplectic. Secondly there are *Lagrangian* submanifolds  $L$ . These are manifolds of half the dimension of  $M$  on which the form  $\omega$  vanishes identically. Thus  $\omega(v, w) = 0$  for every pair of vectors tangent to  $L$ . For example, in a cotangent bundle the fibers  $T_q^*X$  as well as the zero section are Lagrangian. Moreover, the graph  $\{(x, \sigma(x)) : x \in X\}$  of the 1-form  $\sigma$  is Lagrangian if and only if  $\sigma$  is closed. More generally, there are isotropic and coisotropic submanifolds which again are specially situated in terms of the symplectic form.

## §2 The beginning of the modern theory

In this section we discuss developments in the two main strands of symplectic geometry.

### §2.1 Almost complex geometry

For any symplectic manifold  $(M, \omega)$  there is a contractible family of Riemannian metrics on  $M$  associated to  $\omega$ , which are constructed via  $\omega$ -compatible almost complex structures  $J$ . Here  $J$  is an automorphism

$$J : TM \rightarrow TM, \quad J^2 = -\text{Id}$$

that turns the tangent bundle  $TM$  into a complex vector bundle. The compatibility conditions are:

$$\omega(x, y) = \omega(Jx, Jy), \quad \text{and} \quad \omega(x, Jx) > 0 \text{ for all } x \neq 0.$$

They imply that the bilinear form

$$g_J : \quad g_J(x, y) = \omega(x, Jy)$$

is a Riemannian metric (i.e. symmetric and positive definite.) For each  $\omega$  the set of such  $J$  is nonempty and contractible.

**Examples**

- The standard almost complex structure  $J_0$  on  $\mathbf{R}^{2n}$  given by

$$J_0 \left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial y_j}, \quad J_0 \left( \frac{\partial}{\partial y_j} \right) = -\frac{\partial}{\partial x_j}$$

is compatible with  $\omega_0$ .

- The almost complex structure  $J$  induced by the complex structure on a Kähler manifold is compatible with the Kähler form.

An almost complex structure  $J$  on  $M$  is said to be integrable if it comes from an underlying complex structure on the manifold (as in the Kähler case.) There is a tensor  $N_J$  called the Nijenhuis tensor that is made out of the first derivatives of  $J$  and vanishes if and only if  $J$  is integrable. If the dimension of  $M$  is  $> 2$ ,  $N_J$  is usually nonzero. However in the 2-dimensional case all  $J$  are integrable.

Note that the conditions for a manifold to have an almost complex structure are topological in nature and not very demanding. For example every compact 4-manifold  $M$  that satisfies a simple cohomological condition has an almost complex structure  $J$ . However the conditions for the existence of an integrable  $J$  or of a symplectic structure and its corresponding  $J$  are much deeper and are not yet well understood. In some sense the passage from Kähler to symplectic geometry is analogous to the passage from linear to nonlinear geometry that was mentioned above as one aspect of the “local to global” principle. In particular, many (but not all) global features of Kähler manifolds persist in the symplectic setting.

**$J$ -holomorphic curves** Given any symplectic submanifold  $Q$  of  $(M, \omega)$  one can find an  $\omega$ -compatible  $J$  on  $M$  that restricts to an almost complex structure on  $Q$ , i.e.  $J(TQ) = TQ$ . In particular, if  $Q = C$  is 2-dimensional it will be a complex submanifold with respect to any  $J$  such that  $J(TC) = TC$ . Such  $C$  are called  *$J$ -holomorphic curves*. It was the great insight of Gromov [G] to realise that in symplectic geometry these are the correct replacement for geodesics.

It is often useful to parametrize them rather than to consider them as submanifolds. Because the restriction of  $J$  to  $C$  is integrable, there always is a Riemann surface  $(\Sigma, j)$  that maps onto  $C$  by a map  $u : \Sigma \rightarrow M$  that satisfies the generalized Cauchy–Riemann equation:

$$du \circ j = J \circ du.$$

(Here  $j$  is the complex structure on the Riemann surface.) It is not hard to show that the image  $C = u(\Sigma)$  is a minimal surface in  $M$  with respect to the metric  $g_J$ , so the analogy with geodesics is not far fetched.

There is a very nice theory of these curves. Because the Cauchy–Riemann operator is elliptic, they lie in the zero set of a Fredholm operator and so for generic  $J$  form a family (or moduli space) of dimension equal to the index of

the operator. Moreover, although these moduli spaces are not usually compact, they have compactifications (formed by stable maps) whose structure is now well-understood.

As Witten has often pointed out (see for example his 1998 Gibbs lecture [W]), one important consequence of replacing the points that make up the configuration spaces of classical physics by strings (which in the closed case are just circles) is that the time line of a point — usually identified with the real line  $\mathbf{R}$  — is replaced by the time line of a string, which is a cylinder  $S^1 \times \mathbf{R}$ . This cylinder can be identified with the quotient  $\mathbf{C}/\mathbf{Z}$  of the complex plane  $\mathbf{C}$  by a translation and so has a natural complex structure. Thus the passage to string theory involves replacing  $\mathbf{R}$  by  $\mathbf{C}$  (or  $\mathbf{C}/\mathbf{Z}$ ), and so going from a geometry in which 1-dimensional objects such as geodesics are of paramount importance to one in which 2-dimensional objects such as  $J$ -holomorphic curves are the crucial elements. It is no accident that many of the new ideas that have come into mathematics from physics (such as quantum cohomology and mirror symmetry) involve  $J$ -holomorphic curves in an essential way.

## §2.2 Hamiltonian dynamics

We mentioned above that the nondegeneracy of the symplectic form  $\omega$  is equivalent to the condition that there is a bijective correspondence  $X \mapsto \iota_X \omega$  between vector fields and 1-forms. A vector field  $X$  is said to be *symplectic* if its flow<sup>1</sup>  $\phi_t^X : M \rightarrow M$  consists of symplectomorphisms: that is, if

$$(\phi_t^X)^* \omega = \omega, \quad \text{for all } t.$$

The crucial importance of the closedness condition imposed on  $\omega$  is that it implies that the *symplectic* vector fields correspond precisely to the *closed* 1-forms. In particular, every function  $H$  on  $M$  gives rise to a vector field  $X_H$  via the correspondence

$$dH = \iota_{X_H} \omega$$

and hence to a flow  $\phi_t^H$  on  $M$  called the Hamiltonian flow of  $H$ . Because  $dH(X_H) = \omega(X_H, X_H) = 0$ , the orbits  $\phi_t^H(p), t \in \mathbf{R}$  of this flow lie entirely in the level sets  $H = \text{const}$  of the function (or Hamiltonian)  $H$ . Indeed in Hamilton's formulation the motion of a classical mechanical system is precisely a Hamiltonian flow on  $\mathbf{R}^{2n}$  in which  $H$  is the energy of the system, and so we recover the fact that energy is conserved in these systems.

There are many consequences of the existence of these flows. One is that the group  $\text{Symp}(M)$  of symplectomorphisms of  $(M, \omega)$  is always infinite-dimensional. Another is that one can define a Lie bracket (called the Poisson bracket) on the

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<sup>1</sup>The flow of  $X$  is a family of maps  $\phi_t^X : M \rightarrow M, t \in \mathbf{R}$  such that the path  $\phi_t^X(p), t \in \mathbf{R}$  is everywhere tangent to the vector field  $X$ . Here we are assuming that  $M$  is compact so that the flow is defined for all time  $t \in \mathbf{R}$ .

space of smooth functions on  $M$  by setting:

$$\{F, H\} = \omega(X_F, X_H) = \sum_i \frac{\partial F}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{\partial F}{\partial y_i} \frac{\partial H}{\partial x_i},$$

where the last formula is its expression in the coordinates on  $\mathbf{R}^{2n}$ . (This begins to hint at the existence of interesting algebraic structures in symplectic geometry.) Another interesting geometric fact is that the path (or orbit)  $\{\phi_t^H(p), t \in \mathbf{R}\}$  traced out by the flow (but not its parametrization) depends only on the level set  $\{H = H(p)\}$  through the point  $p$ . In fact, any hypersurface (codimension 1 submanifold)  $Q$  in a symplectic manifold has a so-called characteristic foliation — a family of paths on it that always point along the null directions of the restriction  $\omega|_Q$  of  $\omega$ . This gives a completely geometric definition of the flow orbits. However, just as in the theories of classical mechanics where, for example, light rays travel along the shortest path, it is also possible to characterize the flow orbits as minima (or more generally as critical points) of a suitable action functional on a suitable space of paths in  $M$ . This is the simplest instance of the relevance of variational methods in symplectic geometry.

**Symplectic capacities** The geometry of the characteristic foliation has much significance. For example, suppose that  $S$  is the boundary of a generic convex region  $R_S$  in  $\mathbf{R}^{2n}$ . Then only a finite number of the flow lines in its characteristic foliation are circles (rather than infinite arcs.) The size of each such circle  $\gamma$  (or closed leaf, as it is usually called) is given by its *period*

$$\mathcal{P}(\gamma) = \int_D \omega_0$$

where  $D$  is any 2-disc  $\mathbf{R}^{2n}$  with boundary  $\gamma$ . The minimum of these periods is called the *symplectic capacity* of the convex region  $R_S$ .<sup>2</sup> A fundamental result due to Ekeland–Hofer (and, in a different form, to Eliashberg) is that a diffeomorphism  $\phi$  is symplectic (i.e.  $\phi^*(\omega) = \omega$ ) if and only if it preserves the symplectic capacity of all sufficiently small convex regions  $R_S$ .

**Lagrangian submanifolds** The above remarks indicate how codimension 1 submanifolds are relevant to Hamiltonian dynamics. I would now like to argue that Lagrangian submanifolds are equally relevant. As a first illustration, consider the product manifold  $(M \times M, -\omega \times \omega)$ . Any diffeomorphism  $\phi : M \rightarrow M$  is determined by its graph

$$gr(\phi) = \{(x, \phi(x)) : x \in M\},$$

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<sup>2</sup>Another essentially equivalent definition of symplectic capacity was formulated earlier by Gromov [G].

and it is easy to check that  $\phi$  is a symplectomorphism if and only if  $gr(\phi)$  is Lagrangian. Using this, we can describe a second way of generating a symplectomorphism of  $M$  from a function  $F : M \rightarrow \mathbf{R}$  provided that  $F$  and its first and second derivatives are sufficiently small. When  $M = \mathbf{R}^{2n}$ , it is related by means of the Hamilton–Jacobi equation to the flow  $\phi_t^H$  constructed before.

This construction relies on Weinstein’s Lagrangian neighborhood theorem that states that any Lagrangian submanifold  $L$  has a neighborhood  $\mathcal{N}(L)$  that is symplectomorphic to a neighborhood  $\mathcal{N}(L_0)$  of the zero section in the cotangent bundle  $(T^*L, \Omega_{can})$ . Let us apply this when  $L$  is the diagonal in  $(M \times M, -\omega \times \omega)$ . Now suppose given a function  $F$  on  $M$ . As mentioned earlier, the graph of the corresponding closed 1-form  $dF$  is a Lagrangian submanifold of  $T^*M$ , which will lie in the neighborhood  $\mathcal{N}(M_0)$  if  $dF$  is sufficiently close to 0. In this case, it will give rise to a Lagrangian submanifold of the product  $M \times M$  that is close to the diagonal and hence the graph of a symplectomorphism  $\phi_F$ .

There are many other more subtle relations between Lagrangian submanifolds and symplectomorphisms. For example, the Hamiltonian flow generated by  $H : M \rightarrow \mathbf{R}$  is said to be *totally integrable* if there are  $n$  independent functions  $F_1 = H, F_2, \dots, F_n$  that Poisson commute, i.e. the Poisson brackets  $\{F_i, F_j\}$  vanish. (Here  $n$  is half the dimension of  $M$ .) The sets given by the  $n$  equations  $F_1 = c_1, \dots, F_n = c_n$  are then Lagrangian submanifolds (for generic choice of the constants  $c_1, \dots, c_n$ ), and if they are compact, they are tori. Thus, an integrable system defines a family of disjoint Lagrangian submanifolds of  $M$ , one through each point, each of which is invariant under the flow. Many important Hamiltonian systems are integrable in this sense. Moreover this notion of integrable system turns up in many unexpected places, including that of mirror symmetry.

### §2.3 Recent advances

Every symplectic structure  $\omega$  determines a volume form  $\omega^n/n!$ , that is, a nonvanishing top-dimensional form that integrates to give a volume. In two dimensions of course,  $\omega$  is simply an area form. In higher dimensions it was suspected long ago that a symplectic structure is much richer than a volume form, but there was no hard evidence of this until the early 1980s, with Eliashberg’s work on symplectic rigidity, the Conley–Zehnder proof of the Arnold conjecture for the torus, Gromov’s proof of the nonsqueezing theorem [G] and Ekeland–Hofer’s development of the theory of symplectic capacities. These results form the core of modern symplectic topology and can all in some sense be thought of as globalizations of properties of linear (or local) symplectic geometry.

We illustrate this by considering symplectic capacities. Let us first look at the linear case. Here we can restrict consideration to ellipsoids  $E$  since they are preserved by linear maps, and, by change of basis, can assume that  $E$  has the form

$$E(a_1, \dots, a_n) = \{(x_i, y_i) : \sum_i \frac{x_i^2 + y_i^2}{a_i^2} \leq 1, \}$$

where  $a_1 \leq \dots \leq a_n$ . If the  $a_i$  are independent over  $\mathbf{Q}$  there are precisely  $n$  simple closed leaves of the characteristic foliation given by the circles of radius  $a_i$  on which all but one of the coordinate pairs vanish. Hence the symplectic capacity of  $E(a_1, \dots, a_n)$  is just  $\pi a_1^2$ . A relatively easy argument in linear algebra now shows that a linear transformation  $\phi_L$  that preserves the capacity of ellipsoids is either symplectic or antisymplectic, i.e.  $\phi_L^*(\omega_0) = \pm\omega_0$ . The corresponding global result is that a diffeomorphism  $\phi$  preserves capacity if and only if  $\phi^*(\omega) = \pm(\omega)$ . This statement is much deeper and most proofs involve considerable analysis, either variational analysis as in Ekeland–Hofer or elliptic analysis (the theory of  $J$ -holomorphic curves) as in Gromov.

### §3 Some very recent results

To end this article, I will briefly describe two very recent results that bring out the point that the deep results in symplectic topology combine both the complex and the Lagrangian aspects of symplectic geometry.

#### §3.1 Donaldson’s work on Lefschetz pencils

Donaldson developed in [D] a completely new way to use the existence of almost complex structures on symplectic manifolds, taking the manifold  $(M, J)$  to be not the target space but rather the domain of the maps considered. He has developed a theory of “almost holomorphic” sections of certain “almost ample” line bundles that imitates the usual theory in the Kähler case so faithfully that he can prove that every closed (i.e. compact and without boundary) symplectic manifold  $M$  admits a symplectic Lefschetz pencil. This is a smooth map

$$f : M - B \longrightarrow S^2$$

from the complement in  $M$  of a submanifold  $B$  of codimension 4 (the base locus) to the two-dimensional sphere  $S^2$  that can best be thought of as a complex version of a Morse function. In other words,  $f$  has only finitely many singular points, and away from these points the fibers  $f^{-1}(p)$  of  $f$  are symplectic submanifolds of  $M$ . In fact they are “almost holomorphic” in a suitable (and strong) sense. The map  $f$  cannot be extended to  $M$  itself because each fiber contains  $B$  in its closure, but it can be extended to the so-called blow-up  $\widetilde{M}$  of  $M$  along  $B$  in which each point  $x$  of  $B$  is replaced by the set of all pairs  $(x, v)$  where  $v$  is a complex normal direction to  $B$  at  $x$ . The resulting map

$$\tilde{f} : \widetilde{M} \longrightarrow S^2$$

expresses  $\widetilde{M}$  as a kind of twisted product over  $S^2$  with some singularities. Since the generic fiber of  $f$  has the same structure, a symplectic manifold of dimension  $2n$  is built up from 2-dimensional surfaces by repeatedly forming these twisted products over  $S^2$ .

It would seem that this result involves only the “almost complex” side of symplectic geometry. However this is not so. It turns out that the only allowed

singularities of  $f$  can be written in terms of suitable local complex coordinates as

$$f(z_1, \dots, z_n) = \sum_i z_i^2,$$

and, if one investigates the structure of this singularity one sees that it is formed from a (generalized) Dehn twist round the (Lagrangian!) zero section of the cotangent bundle  $T^*(S^{n-1})$ . Thus Lagrangian spheres play an essential role in the understanding of these singularities.

This work of Donaldson's is so recent that it is not yet clear exactly what its implications are for the global structure of symplectic manifolds even in the 4-dimensional case. The existence of Lefschetz fibrations is somewhat analogous to the existence of Heegaard splittings for 3-manifolds, in that it allows one to reduce all questions about symplectic 4-manifolds to (hard) questions about the mapping class groups of Riemann surfaces. In particular, in some sense we now know how to construct all symplectic 4-manifolds while the structure of an arbitrary smooth 4-manifold is still far from clear, even in the simply connected case. There is much current research on this topic, mostly using Seiberg–Witten theory: see [FS].

### §3.2 Quantum cohomology

Lastly, we come to the topic of quantum cohomology which is an essential ingredient of one side of the mirror symmetry phenomenon. The quantum cohomology  $QH^*(M)$  of a compact symplectic manifold is additively the same as its usual cohomology  $H^*(M, \mathbf{C})$  apart from the choice of coefficients, but it has a deformed ring structure in which the product  $a * b$  of two classes is defined as follows. Let  $\langle a, b \rangle$  denote the usual pairing

$$\langle a, b \rangle = \int_M a \cup b$$

on cohomology, and PD be Poincaré duality. Then, given  $a, b \in H^*(M, \mathbf{C})$ ,  $a * b$  is the class in  $QH^*(M)$  defined by

$$\langle a * b, c \rangle = \sum_d n_d(\text{PD}(a), \text{PD}(b), \text{PD}(c)) q^d,$$

where  $q$  is a formal variable, the “degree”  $d$  is an element of  $H_2(M, \mathbf{Z})$  and  $n_d(\alpha, \beta, \gamma)$  is the number of isolated  $J$ -holomorphic spheres in  $M$  that meet cycles representing the homology classes  $\alpha, \beta, \gamma$  and have degree  $d$  (i.e. represent the homology class  $d$ ).<sup>3</sup> The numbers  $n_d(\alpha, \beta, \gamma)$  are called Gromov–Witten invariants, and defining exactly what is meant by them is quite a task that has

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<sup>3</sup>The expressions  $q^d$  obey the usual exponential rules  $q^{d+d'} = q^d q^{d'}$ ,  $q^0 = \text{id}$ , and so really the right hand side is an element of the usual cohomology with coefficients in a suitable completion  $\Lambda$  of the group ring of  $H_2(M, \mathbf{Z})$ .  $\Lambda$  is often called a Novikov ring, and one defines  $QH^*(M)$  to be  $H^*(M; \mathbf{Q}) \otimes \Lambda$ .

only recently been completed. (For references, see [M2].) Note that putting  $q = 0$  has the effect of eliminating all contributions from spheres of nonzero degree. But a sphere of zero degree is simply a constant map. So  $n_0(\alpha, \beta, \gamma)$  is the usual triple intersection of these three classes. Hence

$$n_0(\text{PD}(a), \text{PD}(b), \text{PD}(c)) = \langle a \cup b, c \rangle,$$

and so the product  $*$  really is a deformation of the usual cup product.

From the point of view of mathematics, this is a rather strange definition of a product  $a * b$ , and it is not at all clear why it should give rise to an associative multiplication. However, physicists think of these  $J$ -holomorphic spheres as coming from paths (or more generally Feynman diagrams) in the space of strings which provide needed quantum corrections to the usual product. The combinatorial properties of these Feynman diagrams give rise to relations between the numbers  $n_d(\alpha, \beta, \gamma)$  that imply the associativity of  $*$ : see [RT].

Again, this construction apparently involves only the complex geometry of the symplectic manifold  $M$ . However, there is a huge amount of additional structure hidden here. It turns out that the product  $*$  is generated by a formal power series  $F = F(t, q)$  in the following sense. Choose a basis  $\phi_0 = \mathbb{1} \in H^*(M)$ ,  $\phi_1, \phi_2, \dots$  for  $H^*(M)$  where each  $\phi_j$  is homogeneous, i.e. lies in some  $H^i(M)$ , and write an arbitrary element  $t \in H^*(M)$  as  $t = \sum t_i \phi_i$ . Then the Gromov–Witten potential  $F = F(t, q)$  is a formal power series in  $t, q$  such that

$$\langle \phi_i * \phi_j, \phi_k \rangle = \partial_i \partial_j \partial_k F|_{t=0},$$

where  $\partial_i$  denotes the partial derivative  $\partial/\partial t_i$ . More generally, one gets a family of products  $*_t$  by using the above formula but evaluating at a general point  $t$  instead of  $t = 0$ . The fact that the coefficients  $\langle \phi_i *_t \phi_j, \phi_k \rangle$  are partial derivatives implies that they satisfy certain integrability conditions. In fact, one can use them to define a flat connection on the manifold  $\mathcal{H} = H^*(M, \mathbf{C})/H^2(M, 2\pi i\mathbf{Z})$  and then look at the system of differential equations that describe its flat sections: see [Du]. One way of stating the mirror symmetry conjecture [V] is that this set of equations should be equivalent to a different set that is constructed in a quite different way on the mirror manifold  $M'$ . What I want to point out here is that we are now in the dynamical realm and hence in the world of Lagrangian submanifolds and integrable systems.

The ramifications of these ideas are fascinating but too complicated to discuss here. They are fully worked out only in a few cases, such as toric manifolds: see [Gi]. A simple statement that one can make in this connection is that the quantum cohomology of  $M$  can be expressed as the algebra of (formal) functions on a (formal) Lagrangian variety  $L_M$ . In fact this variety is the characteristic variety of the above mentioned system of differential equations that gives the flat sections. I will end with a simple example that was explained to me by Givental.

Consider complex projective space  $M = \mathbf{C}P^n$ . Its usual cohomology ring is a truncated polynomial ring  $\mathbf{C}[p]/\{p^{n+1} = 0\}$ . To calculate its quantum cohomology note that there is a unique line through 2 points. In terms of Gromov–Witten invariants this says that

$$n_A(\mathbf{C}P^{n-1}, [pt], [pt]) = 1,$$

where  $A$  is the homology class of a line. Hence  $p^n * p = q$  and one finds that the quantum cohomology can be identified with the polynomial ring

$$QH^*(\mathbf{C}P^n) = \mathbf{C}[p, q]/\{p^{n+1} = q\}.$$

The “mirror object” to  $\mathbf{C}P^n$  consists of a family  $Y_q$  of complex tori one for each  $q \in \mathbf{C}^*$  together with a family of functions  $f_q : Y_q \rightarrow \mathbf{C}$ .<sup>4</sup> In fact, in terms of the complex coordinates  $\mathbf{u} = (u_1, \dots, u_{n+1})$  on  $\mathbf{C}^{n+1}$ ,

$$Y_q = \{\mathbf{u} : u_1 u_2 \dots u_{n+1} = q\}, \quad f_q = f|_{Y_q} \quad \text{where} \quad f(\mathbf{u}) = \sum_i u_i.$$

The characteristic variety  $L \subset \mathbf{C}^{n+1}$  is simply given by the set of fiber critical points of the  $f_q$ , i.e.  $L \cap Y_q$  is the set of critical points of  $f_q|_{Y_q}$ . We can embed  $L$  into the cotangent bundle  $T^*(\mathbf{C}^*)$  via the map

$$\mathbf{u} \mapsto (q, \lambda), \quad \text{where} \quad df(\mathbf{u}) = \lambda dq.$$

Here  $\lambda$  is a Lagrange multiplier, and an easy calculation shows that

$$\mathbf{u} \in L \cap Y_q \Leftrightarrow u_i = \lambda q \quad \text{for all } i.$$

Thus  $(\lambda q)^{n+1} = q$  on  $L$ . Now identify  $T^*(\mathbf{C}^*)$  with  $\mathbf{C}^* \times \mathbf{C}$  by using the coordinates  $(q, p) \in \mathbf{C}^* \times \mathbf{C}$  where

$$q = e^t, \quad \lambda dq = p dt.$$

Then  $\lambda dq = (\lambda q) dt$  so that  $p = \lambda q$ . Hence in the  $(p, q)$ -coordinates

$$L = \{(q, p) : p^{n+1} = q\},$$

so that the algebra of (polynomial) functions on  $L$  is precisely the quantum cohomology of  $\mathbf{C}P^{n+1}$ . This calculation can be generalised to the case of toric manifolds: see [Gi].

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<sup>4</sup>Here the coordinate  $q$  that describes the variation in complex structure on  $Y_q$  is essentially the same as the variable  $q$  that appears in the formula for the quantum product and keeps track of the homology class of the curves. This fits in with the yoga of mirror symmetry in which the variation of Kähler class on one manifold corresponds to the variation in complex structure on the mirror. Observe also that the function  $f$  should be thought of as the “total equivariant first Chern class” and so is constant in the case of Calabi–Yau manifolds, since their first Chern class vanishes.

### A final note

I'd like to end on a more personal note. When I first started studying mathematics, it was commonly thought that women did not really have it in them to be mathematicians of the first rank, and there were so few exceptions to this rule that they could be ignored. Over the years, specially as more talented women mathematicians have come on the scene with very varied gifts and personalities, perceptions have shifted. However, in the USA as in many other countries, there are still too few women in mathematics for them to be more than just a token presence in many departments. Because of this, constant effort is still needed to encourage young women to enjoy and develop their mathematical talent. There are many people (both women and men) working at this, and I am hopeful that in the not too distant future the gender of a mathematician will simply no longer be an issue.

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